

# HARMONIC AND INVARIANT MEASURES ON FOLIATED SPACES

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**ABSTRACT.** We consider the family of harmonic measures on a lamination  $\mathcal{L}$  of a compact space  $X$  by locally symmetric spaces  $L$  of noncompact type, i.e.  $L \cong \Gamma_L \backslash G/K$ . We establish a natural bijection between these measures and the measures on an associated lamination foliated by  $G$ -orbits,  $\widehat{\mathcal{L}}$ , which are right invariant under a minimal parabolic (Borel) subgroup  $B < G$ . In the special case when  $G$  is split, these measures correspond to the measures that are invariant under both the Weyl chamber flow and the stable horospherical flows on a certain bundle over the associated Weyl chamber lamination. We also show that the measures on  $\widehat{\mathcal{L}}$  right invariant under two distinct minimal parabolics, and therefore all of  $G$ , are in bijective correspondence with the holonomy invariant ones.

## INTRODUCTION

This article explores measures associated to a nonsingular lamination of an arbitrary compact topological space by locally symmetric spaces. (Note that some authors use the term “foliated space” to mean lamination.) Throughout the paper,  $\mathcal{L}$  will denote a lamination of a topological space  $X$  by leaves that are discrete quotients of the same symmetric space,  $G/K$ , of noncompact type and which is equipped with transversely continuously varying smooth metrics. In particular, each leaf of  $\mathcal{L}$  carries a locally symmetric metric of nonpositive sectional curvature. We do not assume that these metrics are uniformized in any particular way. Hence the curvatures, for instance, can vary from leaf to leaf. Throughout we will assume that the space  $X$  supporting  $\mathcal{L}$  is compact.

We consider two different families of measures associated to  $\mathcal{L}$ . On the one hand, we have the measures invariant under the heat diffusion along the leaves of  $\mathcal{L}$  which are called *harmonic measures*. These measures often arise in the study of topological and ergodic properties of a foliation, in part because they generalize the holonomy groupoid invariant measures which may not always exist. On the other hand, we have the class of measures on the natural  $K$ -bundle lamination  $\widehat{\mathcal{L}}$  associated to  $\mathcal{L}$  that are invariant under the right action by a fixed common choice of Borel subgroup of  $B < G$ . In the split case, i.e. when the intersection of every Cartan subgroup with  $K$  is

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trivial, this latter group of measures has a dynamical characterization. They are the measures which are invariant under the combined Weyl chamber flow and horospherical (unipotent) flows of Weyl chambers.

In the case when  $\text{rank}(G) \geq 2$ , there are two competing notions of harmonicity: vanishing under the standard differential geometric laplacian versus the stricter requirement of vanishing under all second order elliptic  $G$ -invariant operators which annihilate constants. However, we will show (see Section 2.1) that in our setting these notions coincide.

Our main result reads as follows:

**Theorem I.** *The canonical projection  $\widehat{\mathcal{L}} \rightarrow \mathcal{L}$  induces a bijective correspondence between the  $B$ -invariant measures on  $\widehat{\mathcal{L}}$  and the  $\Delta_{\mathcal{L}}$ -harmonic measures on  $\mathcal{L}$ .*

When  $\mathcal{L}$  is at least transversally continuous,  $B$ -invariant measures always exist on  $\widehat{\mathcal{L}}$  since  $B$  is an amenable group acting continuously on the compact space  $\widehat{\mathcal{L}}$ . In particular, this gives another proof of the existence of harmonic measures. Theorem I also holds in the case where  $\mathcal{L}$  is only transversely measurable, but then the property of being a harmonic measure becomes more restrictive, and there may not exist any (see Section 2.1).

Naturally, the fiber over a given harmonic measure under the induced canonical projection of measures consists of more than just the unique  $B$ -invariant measure. We will also describe measures in this fiber which are invariant only under the right action by the subgroup  $N \rtimes A$  of  $B = NAM$ . (See Section 1 for definitions.)

*Remark 0.1.* The main result of this article was carried out in the special case of surface laminations, namely when  $G = PSL(2, \mathbb{R})$ , by the second author and Yu. Bakhtin in the two articles [Mar06] and [BM08]. In this paper, besides establishing the results in complete generality, we also manage to significantly shorten the proofs, albeit at the expense of a bit more machinery.

We also establish a couple of other related results along the way. Given a harmonic measure  $\mu$ , in Theorem 4.3 we give an explicit geometric construction for the unique  $B$ -invariant lift  $\widehat{\mu}$  on  $\widehat{\mathcal{L}}$  of any given harmonic measure  $\mu$  on  $\mathcal{L}$ .

Finally in the last section, we will provide a short proof of the following related result.

**Theorem II** ([MM09]). *There is a natural bijective correspondence between  $G$ -invariant probability measures on  $\widehat{\mathcal{L}}$  and invariant transverse measures on  $\mathcal{L}$ .*

In Section 1 we review those concepts that we will need from the theory of semisimple Lie groups and locally symmetric spaces. In Section 2 we review laminations by symmetric spaces and harmonic measures. Section 3

is devoted to the proof of Theorem I. In Section 4 we give an explicit construction of a measure on  $\widehat{\mathcal{L}}$  invariant under a minimal parabolic subgroup that projects onto a given harmonic measure on  $\mathcal{L}$ . Finally, in Section 5 we give an interesting application along with the proof Theorem II.

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#### 1. THE ACTION OF $G$ AND THE WEYL CHAMBER FLOW AND UNIPOTENT FLOWS

Let  $G$  be a semisimple Lie group of noncompact type with Lie algebra  $\mathfrak{g}$ . Choose a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ . Let  $\Lambda$  be the corresponding set of nonzero roots giving the corresponding root space decomposition  $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha$ , where  $\mathfrak{h}$  is identified with the 0-root space  $\mathfrak{g}_0$ . Moreover we have the associated Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ . We denote the Cartan subspace of  $\mathfrak{p}$  by  $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{h}$  and its centralizer by  $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{h}$ .

Let  $K$  be the maximal compact subgroup of  $G$  whose Lie algebra is  $\mathfrak{k}$ . The projection  $G \rightarrow G/K$  is a diffeomorphism when restricted to the subset  $\exp(\mathfrak{p}) \subset G$ . Moreover under the identification of  $G$  with the group of isometries  $\text{Iso}(G/K)$ ,  $K$  can be identified with the isotropy subgroup  $K_p$  of some point  $p \in G/K$ . Doing so, we can describe any geodesic through  $p$  as  $\exp(th) \cdot p$  where  $h \in \mathfrak{p}$ . Moreover,  $h$  belongs to at least one maximal abelian subalgebra living in  $\mathfrak{p}$ . By varying  $p$ , and hence the choice of maximal compact  $K = K_p$  and the corresponding Lie group and Lie algebra decompositions, all geodesics can be described as orbits under left actions by one parameter subgroups.

We let  $A = \exp(\mathfrak{a})$ , and denote the centralizer and normalizer of  $A$  in  $K$  by  $M$  and  $M'$  respectively. They both have the same Lie algebra  $\mathfrak{m}$  introduced above. Moreover,  $M$  is normal in  $M'$  and  $W = M'/M$  is the corresponding Weyl group. Choosing a positive Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$ , we obtain a corresponding set of positive roots  $\Lambda^+$ . We write the maximal nilpotent subalgebra of  $\mathfrak{g}$  corresponding to this decomposition as  $\mathfrak{n} = \bigoplus_{\alpha \in \Lambda^+} \mathfrak{g}_\alpha$ . The corresponding maximal unipotent subgroup will be denoted by  $N < G$ . The corresponding Borel subgroup is the maximal solvable subgroup  $B = MAN < G$  which corresponds to the maximal solvable subalgebra  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ .

Recall that a semisimple Lie group has the following Bruhat decomposition  $G = \bigcup_{w \in W} BwB$  where  $W < K$  is the isomorphic copy of the Weyl group generated by reflections across chamber walls of  $\mathfrak{a}$ . Since the Weyl

group is simply transitive on chambers,  $M$  can also be described as the stabilizer of any single Weyl chamber of  $\mathfrak{a}$ . Therefore we can identify  $G/M$  as the bundle of Weyl chambers of the symmetric space  $G/K$ . In the case that  $G$  has rank 1,  $G/M$  is just the (total space of the) unit tangent bundle. Note that by the Iwasawa decomposition  $G = KAN = NAK$ , we may identify  $G/K$  with a solvable subgroup  $NA$ , and the Weyl chamber bundle  $G/M$  with  $NA(K/M)$ . For instance if  $G = PO(n, 1) = \text{Iso}(\mathbb{H}^n)$  then  $K/M = SO(n)/SO(n-1) \cong S^{n-1}$  so  $G/M = NAK/M \cong \mathbb{H}^n \times S^{n-1}$  is naturally identified with the unit tangent bundle  $T^1\mathbb{H}^n$ .

Note that the right translation action of  $M$  on both  $G/M$  and  $G/K$  is trivial. For  $G$  of arbitrary rank, the geodesic flow naturally generalizes to the Weyl chamber flow by the right action of  $A$  on  $G/M$ . (Of course the geodesic flow on  $T^1(G/K)$  exists in higher rank as well, but it is not induced by a right action.) Since  $M$  is the centralizer of  $A$  this action makes sense and commutes with the quotient of the action of  $M$  on  $G$ . In other words, for any  $a \in A$ , we have  $gMa = gaM$ .

This action of  $A$  on  $G/M$  is simply the quotient under  $M$  of the right action of  $A$  on  $G$ . Dissimilarly, the right action of  $N$  on  $G$  does not descend to an action on  $G/M$  since  $M$  does not, in general, centralize  $N$ , and hence the action of  $N$  does not commute with  $M$ . More explicitly, the map  $gM \mapsto gnM$  is not well defined since for any  $k \in M$  we have  $gM = gkM$ , but  $gnM \neq gknM$ .

We will need to introduce a few other notions. The Furstenberg boundary of  $G/K$  is the space  $G/B$ . From the Iwasawa decomposition, we naturally see that  $G/B = K/M$  is the compact space of Weyl chambers of  $G$  based at  $e \in G$ . Since  $N < B$ , the left action by this subgroup fixes points in  $G/B$ . Every Weyl chamber  $C \subset G/K$  asymptotically approaches a Weyl chamber  $C'$  which passes through  $eK$ . More specifically  $\partial_\infty C = \partial_\infty C' \subset \partial_\infty G/K$ . Hence the Weyl chamber  $C$  at infinity can be identified with some point  $[C] = [C'] = q \in G/B$ . The action of  $N$  on  $G/B$  fixes the canonical point  $eB \in G/B$  and each element of  $N$  fixes no other point. Moreover, for any element  $q \in G/B$  there is an element  $k \in K$ , unique modulo  $M$ , such that  $kq = eB \in G/B$ .

## 2. LAMINATIONS BY SYMMETRIC SPACES

We say  $\mathcal{L}$  is a  $C^{r,s,u}$  *lamination* for  $r, s, u \in [0, \infty] \cup \{\omega\}$  with  $s \leq r$  if there is a separable, locally compact space  $X$  that has an open covering  $\{E_i\}$  and an atlas  $\{(E_i, \varphi_i)\}$  satisfying:

- (1)  $\varphi_i : E_i \rightarrow D_i \times T_i$  is a homeomorphism, for some open ball  $D_i$  in  $\mathbb{R}^n$  and a topological space  $T_i$ , and
- (2) the coordinate changes  $\varphi_j \circ \varphi_i^{-1}$  are of the form  $(z, t) \mapsto (\zeta(z, t), \tau(t))$ , where each  $\zeta$  is  $C^r$  smooth in the  $z$  variable with all derivatives up to order  $s$  varying in a  $C^u$  smooth way in the  $t$ -variable.

This last condition says that the sets of the form  $\varphi_i^{-1}(D_i \times \{t\})$ , called *plaques*, glue together to form  $n$ -dimensional  $C^r$ -manifolds that we call *leaves* which vary in a  $C^u$  way up to order  $s$ . Said differently, the  $s$ -jet transversal holonomies are  $C^u$ .

We will also be interested in considering measurable  $C^{r,s}$  laminations, which are  $C^r$  smooth in the  $z$  variable with all derivatives up to order  $s$  varying measurably in the  $t$  variable. As a shorthand, a continuous (resp. measurable) lamination means a  $C^{\infty,\infty,0}$  (resp.  $C^{\infty,\infty}$  measurable) lamination. In what follows we will suppress the mention of regularity, as the needed regularity is generally obvious from the context.

In any class of maps between foliated spaces we can define the subclass of laminated maps to be those which carry leaves to leaves. A laminated fibration (or fibre bundle)  $E \rightarrow X$  is a fibration map between the laminated spaces  $E$  and  $X$  which is a laminated map and whose local product structure is compatible with the local product structure of the laminations. As an example, given a lamination  $\mathcal{L}$  by  $C^1$  manifolds, we can define the tangent bundle lamination  $T\mathcal{L}$  to be the lamination which is locally the product of the tangent space to the leaves with the same transversals as  $\mathcal{L}$ . Similarly, an assignment of metrics to the leaves of  $\mathcal{L}$  corresponds to a section of the laminated bundle  $S^2T^*\mathcal{L} \rightarrow \mathcal{L}$  of symmetric 2-tensors on the cotangent bundle  $T^*\mathcal{L}$ .

We will say that a lamination  $\mathcal{L}$  is a *locally symmetric lamination* if the disks  $D_i$  are open subsets of  $G/K$  and the maps  $\zeta$  are isometries in the  $z$  variable. When the underlying space  $X$  is compact, each leaf  $L$  of  $\mathcal{L}$  will be a complete locally symmetric space, and therefore each of the form  $\Gamma_L \backslash G/K$  for some discrete subgroup  $\Gamma_L < G$ . Each leaf on a locally symmetric lamination  $\mathcal{L}$  will be endowed with the locally symmetric metric resulting from pulling back by the chart. The transverse regularity of  $\mathcal{L}$ , being the same as that of the isometries  $\zeta(\cdot, t)$ , will be the transverse regularity of the identification of the leaves with the locally symmetric spaces  $\Gamma_L \backslash G/K$ .

For any locally symmetric lamination  $\mathcal{L}$ , we can construct its associated  $K$ -bundle lamination  $\widehat{\mathcal{L}}$  as follows. First construct the principal  $K$ -bundle  $\widehat{X} \xrightarrow{\pi} X$  over  $X$  whose fiber over a point  $x \in L$ , identified with the double coset  $\Gamma_L g_x K$  for some  $g_x \in G$  is simply again  $\Gamma_L g_x K$ , but now viewed as a set of left cosets.

The bundle charts  $(\widehat{E}_i, \widehat{\varphi}_i)$  with  $\widehat{\varphi}_i : \widehat{E}_i \rightarrow K \times D_i \times T_i$  cover the chart  $(E_i, \varphi_i)$  from  $\mathcal{L}$  in the sense that the following diagram commutes,

$$\begin{array}{ccc} \widehat{E}_i & \xrightarrow{\widehat{\varphi}_i} & K \times D_i \times T_i \\ \pi \downarrow & & \downarrow \pi_0 \times \text{id} \\ E_i & \xrightarrow{\varphi_i} & D_i \times T_i \end{array}$$

where  $\pi_0$  is the natural projection and  $\pi$  is the restricted bundle map.

The coordinate changes  $\widehat{\varphi}_j \circ \widehat{\varphi}_i^{-1}$  are of the explicit form  $(k, z, t) \mapsto (\eta(k, z, t), \zeta(z, t), \tau(t))$  where  $\zeta$  and  $\tau$  are as above, and  $\eta$  is an isometry of  $K$ . Finally we form the lamination  $\widehat{\mathcal{L}}$  of the underlying space  $\widehat{X}$  by taking the lifts of the leaves  $L$  of  $\mathcal{L}$  under  $\pi$  to obtain leaves  $\widehat{L}$  each of dimension  $n + \dim(K)$ . Strictly speaking, the charts  $(\widehat{E}_i, \widehat{\varphi}_i)$  do not form a set of lamination charts for  $\widehat{\mathcal{L}}$  since  $K$  is not generally a topological disk. This can easily be taken care of by finitely subdividing the charts  $(\widehat{E}_i, \widehat{\varphi}_i)$  further into restricted charts  $(\widehat{U}_\alpha, \widehat{\phi}_\alpha)$  where for each new index  $\alpha$ ,  $\widehat{U}_\alpha \subset \widehat{E}_i$  for some  $i = i(\alpha)$  and  $\widehat{\phi}_\alpha : \widehat{U}_\alpha \rightarrow \widehat{D}_\alpha \times T_i$  is a homeomorphism equal to the restriction of  $\widehat{\varphi}_i$  and where  $\widehat{D}_\alpha \subset K \times D_i$  is a topological ball.

We endow each leaf  $\widehat{L}$  of  $\widehat{\mathcal{L}}$  with the unique right  $K$ -invariant metric which projects to the metric on the leaf  $L = \Gamma_L \backslash G/K$  of  $\mathcal{L}$ . Thus each leaf  $\widehat{L}$  is isometric to  $\Gamma_L \backslash G$  and the regularity of  $\widehat{\mathcal{L}}$  will be identical to that of  $\mathcal{L}$ .

What is less obvious is that there is a consistent right action of  $G$  on  $\widehat{\mathcal{L}}$ . In other words, the identification of each leaf  $L$  with  $\Gamma_L \backslash G/K$  can be done consistently with the foliation structure. This is the result of Proposition 2.5 of [Zim88].

The transverse regularity of the isometries  $\eta$  and  $\zeta$  guarantees that the right  $G$  action on each leaf of  $\mathcal{L}$  has the same transverse regularity. In other words, action can be defined on each leaf using the identification  $L = \Gamma_L \backslash G/K$ , and the compatibility and regularity of charts guarantee the regularity, locally and hence globally, of the action. Note that this does not imply that the  $\Gamma_L$  vary continuously, and indeed they generally do not.

Lastly, we note that we can similarly define the lamination  $\check{\mathcal{L}}$  whose leaves are of the form  $L = \Gamma_L \backslash G/M$  as bundle over  $\mathcal{L}$ . Since the metric on the leaves of  $\widehat{\mathcal{L}}$  are both right and left  $K$ -invariant, and  $M < K$ , we could also realize  $\check{\mathcal{L}}$  as a leafwise metric quotient of  $\widehat{\mathcal{L}}$ .

**Examples 2.1.** (1) By a theorem due to Candel (see [Can93]), any continuous lamination by Riemannian surfaces for which the laminated Euler characteristic is negative admits a globally continuous conformal change of leafwise metric to ones of constant negative curvature. The resulting lamination will be a symmetric space lamination locally by  $\mathbb{H}^2 = \mathrm{PSL}(2, \mathbb{R})/\mathrm{SO}(2)$ . Note that any closed leaf of genus  $g > 1$  admits a  $6g - 6$  dimensional family, the Teichmüller space, of conformally inequivalent metrics. Each such metric can be extended to a continuous family of metrics on the foliation by a partition of unity argument. In this case, Candel's theorem produces a corresponding family of pairwise metrically distinct hyperbolic laminations.

(2) If we have a group action by a semisimple group  $G$  on  $X$  for which the stabilizer of each point is conjugate to  $K$ , then the orbits form a symmetric space lamination. The metrics on the leaves, each one

homeomorphic to  $G/K$ , are induced from the Killing form on  $G$ . The transversal regularity of the foliation coincides with the (transversal) regularity of the action.

- (3) Another source of laminations comes from suspensions of a discrete group action. Let  $\Gamma$  be a discrete group acting freely and properly discontinuously on a manifold  $M$  so that  $\Gamma \backslash M$  is a compact (Hausdorff) manifold. Suppose  $\Gamma$  also admits an action on a compact space  $Y$  by homeomorphisms, which we indicate as a right action. Then we can form the compact space  $X = Y \times_{\Gamma} M = (Y \times M) / \sim$  where  $(y, z) \sim (y\gamma, \gamma z)$  for all  $\gamma \in \Gamma$ . The space  $X$  carries the structure of a compact lamination with a global transversal  $Y$  whose leaves are the images of  $M$  in the quotient space  $X$ . If  $M = G/K$  then the leaves are locally symmetric spaces with each  $\Gamma_L < \Gamma$ .

Throughout the remainder of this paper,  $\mathcal{L}$  will always denote a continuous (that is  $C^{\infty, \infty, 0}$ ) locally symmetric lamination on a compact space  $X$ , unless otherwise specified.

**2.1. Harmonic measures.** Each leaf  $L$  of  $\mathcal{L}$ , being a Riemannian manifold, has a Laplace-Beltrami operator  $\Delta_L$ . If  $f : X \rightarrow \mathbb{R}$  is a function of class  $C^2$  in the leaf direction and  $x \in \mathcal{L}$ , we define  $\Delta_{\mathcal{L}} f(x) = \Delta_L f|_L(x)$ , where  $L$  is the leaf passing through  $x$  and  $f|_L$  is the restriction of  $f$  to  $L$ . For some purposes it is convenient to extend  $\Delta_{\mathcal{L}}$  to the functions which leafwise belong to the Sobolev space  $H^2(L, \mathbb{R})$ . By the standard regularity theory for elliptical operators, such an extension would not enlarge the class of  $L^2$  harmonic functions.

Let  $C^2(\mathcal{L})$  denote the continuous leafwise  $C^2$  functions on  $\mathcal{L}$ . We say that a Radon measure  $\mu$  is *harmonic* if  $\Delta_{\mathcal{L}} \mu = 0$  weakly; i.e. if  $\int \Delta_{\mathcal{L}} f d\mu = 0$  for all  $f \in C^2(\mathcal{L})$ .

If we only assume that  $\mathcal{L}$  is transversely measurable ( $C^{\infty, \infty}$  measurable), then we will call  $\mu$  harmonic only if it satisfies the more stringent condition that  $\int \Delta_{\mathcal{L}} f d\mu = 0$  for all Borel measurable leafwise  $C^2$  functions  $f$ . The more stringent requirement is necessary to guarantee that all compactly supported functions on a given leaf are included as test functions.

*Remark 2.2.* Since a general measurable leafwise  $C^2$  function  $f$  will be unbounded, we cannot guarantee the existence of local extrema for  $f$ . The latter is a key step in the application of the Hahn-Banach theorem to show existence of harmonic measures in the transversely continuous case (see Lemma 3.4 and Theorem 3.5 of [Can03]). Consequently, we cannot guarantee the existence of harmonic measures in the transversely measurable setting.

Let  $d\text{vol}$  signify the Riemannian volume measure on each leaf  $L$ . This volume is the projection of the Haar measure of  $G$  to  $\Gamma_L \backslash G/K$ . It is known (e.g. Proposition 5.2 of [Can03]) that every harmonic measure  $\mu$  on  $\mathcal{L}$  can be decomposed locally on a flow box as  $d\mu(x, t) = h(x, t) d\text{vol}(x) \times d\sigma(t)$



where  $\sigma$  is a measure on the transversal  $T$ , and for  $\sigma$ -almost every  $t \in T$ ,  $h(\cdot, t)$  is an harmonic function on the corresponding plaque. Moreover, this decomposition is not unique, but if  $d\mu(x, t) = h'(x, t) \operatorname{dvol}(x) \times d\sigma'(t)$  is another decomposition, then  $h(x, t) = \left(\frac{d\sigma'}{d\sigma}(t)\right) h'(x, t)$ . In other words,  $h$  is well defined up to a positive constant multiple on each plaque.

As mentioned in the introduction, for the case of higher rank symmetric spaces,  $G/K$ , there appears a potentially distinct tradition governing what it means for a function, and by extension a measure, to be harmonic. We have up to this point followed the common differential geometric notion that a harmonic function is one which vanishes under the standard Laplacian, or equivalently possesses the usual spherical averaging property. However the most common tradition in the special setting of higher rank locally symmetric manifolds requires vanishing under the larger family of all  $G$ -invariant second elliptic order operators without constant term. This corresponds to having the self-averaging property on all  $K$ -orbits which are proper subsets of spheres. In higher rank, the Laplacian is no longer the unique such operator. Nevertheless, for bounded functions, it follows from the seminal work of Furstenberg [Fur63] that these two notions coincide. We now show that essentially the same proof yields the coincidence of these two points of view in our setting as well.

**Proposition 2.3.** *The functions  $h(\cdot, t)$  and measure  $\mu$  defined above are harmonic with respect to all  $G$ -invariant second order elliptic operators which annihilate constants.*

*Proof.* The statement reduces to showing that, for a given  $h(\cdot, t)$ , the global well-defined extension  $\hat{h}$  along plaques on the holonomy cover of the leaf is harmonic with respect to the other invariant elliptic operators.

This statement is proven in Theorem 4.4 of [Fur63] for the case when  $\hat{h}$  is bounded. However, the boundedness condition for this direction is used in the proof in only one place namely, to show that

$$(2.1) \quad \frac{d}{dt}\bigg|_{t=0} \int_G \tilde{h}(gg') d\mu_t(g') = \tilde{h}(g)$$

where  $\tilde{h}$  is the lift to  $G$  of  $\hat{h}$  and  $\mu_t$  is the heat semigroup for the standard Laplacian on  $G$ . Since this holds for compactly supported functions, we need to show that the tail of the integral vanish, as a function of  $t$ , at least as fast as  $o(t)$ .

On the other hand, Proposition 4.4 of [ACR12] shows that the 1-form  $\eta = d_{\mathcal{F}} \log h$  is bounded via a Harnack-type inequality. This implies that such a function  $\hat{h}$  on the holonomy cover can only grow at most exponentially fast. However, the heat kernel has decay described in equation (4.3) that we analyze at the end. In particular its highest order decay is Gaussian in the distance and  $t$ , which means that the portion of the integral outside of any ball centered at the identity both converges and vanishes faster than any polynomial in  $t$ . Hence we obtain (2.1) and the proposition.  $\square$



*Remark 2.4.* The averaging property of harmonic functions under certain  $K$ -invariant measures  $\eta$  on  $G$  translates into an invariance of a harmonic measure  $\mu$  under convolution by  $\eta$ , namely  $\eta * \mu = \mu$ . Such measures  $\mu$  are called  $\eta$ -stationary, and were introduced and explored in [Fur63b]. While it is more common to consider stationary measures on a boundary for a random walk generated by  $\eta$ , in our current setting harmonic measures are stationary under the measure generating leaf-wise Brownian motion, a point of view that we will explore shortly.

Later on, we will need to generalize the principle behind the above extensions to certain operators on measures. Let  $\mathcal{A}$  and  $\mathcal{A}'$  be two subspaces of  $\mathcal{B}(X)$ , the space of Borel functions on the underlying space  $X$  of  $\mathcal{L}$ , and let  $\mathcal{M}$  and  $\mathcal{M}'$  be two subspaces of signed Borel measures on  $X$ . Given an operator  $D : \mathcal{A} \rightarrow \mathcal{A}'$ , if for any  $\mu \in \mathcal{M}$  there is a unique measure  $D(\mu) \in \mathcal{M}'$  satisfying

$$\int_{\mathcal{L}} f dD(\mu) = \int_{\mathcal{L}} D(f) d\mu \quad \text{for all } f \in \mathcal{A},$$

then we can extend  $D$  to an operator  $D : \mathcal{M} \rightarrow \mathcal{M}'$ . For instance, if  $D$  is (continuous) linear, and bounded by some norm, and  $\mathcal{A}$  is dense in  $C(X)$ , then  $D$  extends to the Radon measures by an application of the Hahn-Banach theorem followed by the Riesz theorem.

### 3. PROOF OF THEOREM I

Let  $p \in G/K$  denote the identity coset of  $K$ . Using the Iwasawa decomposition we may express  $G/K$  as  $N \rtimes A$  where  $A$  and  $N$  are from Section 1. The group  $N = N_\zeta$  is the unipotent group corresponding to the nilradical which fixes some (regular) point  $\zeta \in G/B$  and similarly  $A = A_\zeta$  also fixes  $\zeta$ . Under this identification the volume form on  $G/K$ , which is itself just the projection of the Haar measure on  $G$  to  $G/K$ , coincides with the left Haar measure on  $N_\zeta \rtimes A_\zeta$ . This in turn coincides with the left Haar measure on  $N_\zeta \rtimes A_\zeta$  for any choice of  $x$  and  $\zeta$  under the identification of this simply transitive subgroup with  $G/K$ . However, unlike the semisimple group  $G$  and the compact group  $K$ , the solvable group  $N_\zeta \rtimes A_\zeta$  is not unimodular. The modular function  $\delta$  on  $N_\zeta \rtimes A_\zeta$  may be expressed in terms the Poisson kernel  $\mathcal{K}$ . More precisely,  $\delta(g) = \mathcal{K}(p, gp, \zeta)$  for  $g \in N_\zeta \rtimes A_\zeta$ . Hence we can write the right Haar measure  $m_{N_\zeta \rtimes A_\zeta}^R$  as the product of  $\mathcal{K}(p, gp, \zeta)$  and the left Haar measure  $m_{N_\zeta \rtimes A_\zeta}^L$  on  $N_\zeta \rtimes A_\zeta$ .

Let  $m_{G/B}$  be the projection of Haar measure on  $K$  to  $K/M = G/B$ . (Note that we may make the identification  $K/M = G/B$  since the action of  $K = K_p$ , while fixing  $p$ , acts transitively on the regular points  $\xi \in G/B$  and  $M = M_p^\xi$  is the stabilizer of the regular point  $\xi$  in  $K_p$ .) The Furstenberg boundary,  $G/B$ , equipped with  $m_{G/B}$  is naturally isomorphic as a measured  $G$ -space to the Poisson boundary of  $G/K$  for the Brownian motion, and the corresponding Poisson kernel is  $\mathcal{K}$  (e.g. see[GJT98]).

In the flow box decomposition for  $\mu$  given in the last section, each function  $h(\cdot, t)$  is a positive harmonic function on the plaque corresponding to  $t$ , and therefore admits a unique extension to the plaques along any simple path in the leaf. Whenever a leaf of the lamination has trivial holonomy, then all extensions agree to give a unique globally defined harmonic function on  $\Gamma_L \backslash G/K$ . In particular, this holds on the holonomy cover  $L'$  of any leaf  $L$ . Without loss of generality, in what follows we may assume that  $L$  has trivial holonomy as we will only be using  $h$  locally. Passing to the universal cover of the leaf, we again denote the  $\Gamma_L$  invariant harmonic function on  $\tilde{L}$  by  $h$ . As a positive harmonic function on  $G/K$  it admits a unique integral representation on the minimal Martin boundary  $\partial_{\min} G$  of  $\tilde{L} \cong G/K$ , namely  $h(x, t) = \int_{\partial_{\min} G} \mathcal{K}(p, x, \xi) d\mu_t(\xi)$  (see I.7.9 of [BJ06]). Here  $\mu_t$  is a Borel measure associated uniquely to  $h(\cdot, t)$  and  $\mathcal{K}(p, x, \xi)$  is the Martin Kernel associated to the point  $\xi \in B_M$  for a fixed basepoint  $p \in G/K$ .

In general,  $\partial_{\min} G$  can be written as the disjoint union,

$$\partial_{\min} G = \coprod_B \overline{\mathfrak{a}_B^+(\infty)}$$

where  $B$  runs over all minimal parabolic subgroups of  $G$  and  $\overline{\mathfrak{a}_B^+(\infty)}$  is the closure of the geodesic boundary of the corresponding positive Weyl chamber (see I.7.18 of [BJ06]). In the case that  $G$  has rank greater than one,  $\partial_{\min} G$  is strictly larger than the Poisson Boundary  $G/B$ , as a topological space. The latter naturally sits inside  $\partial_{\min} G$  as the Furstenberg boundary  $\partial_F G \subset \partial_{\min} G$ , the union of a single point representing the barycenter of each simplex  $\mathfrak{a}_B^+(\infty)$ . If  $h(\cdot, t)$  were globally bounded then  $d\mu_t = f_t dm_{G/B}$  for a unique positive function  $f_t$  on  $G/B$  representing the limit values of  $h(\cdot, t)$  along random walks.

It turns out that the  $h(\cdot, t)$  need not be a globally bounded harmonic function. On the other hand, the  $h(\cdot, t)$  are not quite arbitrary since they are leafwise Jacobians of a finite measure on a compact foliated space which places a restriction on their overall growth. As explained in Section 6 of [Can03], the diffusion operator generating the harmonic measure only depends on the Brownian motion on each leaf. Since every Brownian path almost surely converges to a point in  $\partial_F G$  (see 6.2 of [GJT98]), it follows that the lifted local harmonic functions  $h(\cdot, t)$  are representable by measures on the Poisson boundary as,

$$h(x, t) = \int_{\partial_F G} \mathcal{K}(p, x, \xi) d\mu_t(\xi),$$

where we now consider  $\mu_t$  to be a measure with support on  $\partial_F G \cong G/B$ .

Since  $\mathcal{K}(p, \gamma^{-1}x, \xi) = \mathcal{K}(\gamma p, x, \gamma\xi) = \mathcal{K}(p, \gamma^{-1}p, \xi)\mathcal{K}(p, x, \gamma\xi)$ , the  $\Gamma_L$  invariance of  $h$  implies that  $\frac{d\gamma_*^{-1}\mu_t}{d\mu_t}(\xi) = \mathcal{K}(p, \gamma^{-1}p, \xi)$ . In particular, if  $\Gamma_L$  acts ergodically on  $G/B$  with respect to  $\mu_t$ , such as when the leaf  $L$  has finite volume, then uniqueness of conformal densities implies that  $\mu_t$  is a multiple of  $dm_{G/B}$ . This implies that  $h$  is the constant function  $\|\mu_t\|$ .

*Remark 3.1.* An alternate approach to the representation of harmonic functions arises from using hyperfunctions in the sense of Sato. (These form a class of generalized distributions.) It was shown in [KKM78] that every harmonic function can also be represented via a Poisson formula applied to a hyperfunction on  $G/B$ . In particular, the measures  $\mu_t$  above also arise as hyperfunctions.

Take a Borel measurable section of  $K/M$  into  $K$  represented by  $\xi \rightarrow k^\xi$ . (In particular we have  $k_\xi \zeta = \xi$ .) We first note that we can express  $\text{dvol}$  on  $G/K$  as the projection under  $P : G \rightarrow G/K$  of the left invariant Haar measure on the orbit  $N_\zeta \rtimes A_\zeta \cdot k^\xi$  in  $G$  for any fixed  $\zeta \in G/B$ . This latter measure can be written as the push forward  $k_*^\xi dm_{N_\zeta \rtimes A_\zeta}^L$ . Hence fixing  $\zeta \in G/B$  we may write:

$$\begin{aligned}
 h(x, t) \text{dvol}_L &= \left( \int_{G/B} \mathcal{K}(p, x, \xi) d\mu_t(\xi) \right) \text{dvol}_{G/K} \\
 &= \int_{G/B} \mathcal{K}(p, x, \xi) P_* k_*^\xi dm_{N_\zeta \rtimes A_\zeta}^L d\mu_t(\xi) \\
 (3.1) \quad &= \int_{G/B} \mathcal{K}(k^\xi p, x, k^\xi \zeta) P_* k_*^\xi dm_{N_\zeta \rtimes A_\zeta}^L d\mu_t(\xi) \\
 &= \int_{G/B} P_* k_*^\xi \left( \mathcal{K}(p, x, \zeta) dm_{N_\zeta \rtimes A_\zeta}^L \right) d\mu_t(\xi) \\
 &= \int_{G/B} P_* k_*^\xi dm_{N_\zeta \rtimes A_\zeta}^R d\mu_t(\xi).
 \end{aligned}$$

In order to lift  $\mu$  we first need to choose a family of Borel probability measures  $\{\mu_{x,\xi}\}$ , each supported on its corresponding left coset  $xk^\xi M_p^\zeta = M_{xp}^{x\xi} xk^\xi$  for  $x \in N_\zeta \rtimes A_\zeta$ . (These will correspond to the conditional measures under the disintegration of  $\hat{\mu}$  along  $\mu$ .) We will eventually see that  $\mu_{x,\xi}$  must be chosen to be the copy of the Haar measures on  $xk^\xi M_p^\zeta$ . We will progressively impose conditions on these measures until they are uniquely specified. First, they must be left  $\Gamma_L$  equivariant, by which we mean that  $\gamma_* \mu_{x,\xi} = \mu_{\gamma x, \gamma \xi}$  for all  $\gamma \in \Gamma_L$  where  $L$  is the leaf of  $\mathcal{L}$  containing  $x$ . Decomposing  $\mu$  in each foliated chart, we obtain  $d\mu = d\tau_t \times d\sigma$  where  $\sigma$  is a measure on the transversal  $T$  and  $\tau_t$  is a measure on the plaque passing through  $t \in T$ . Since  $\mu$  is a harmonic measure, the measure  $\tau_t$  locally has the form  $h \text{dvol}_L$  for an  $\Delta_L$ -harmonic function  $h$ , as described above. Usually,  $K$  as an  $M$  bundle over  $K/M$  is not trivial. (Consider, for instance, the case when  $L = \mathbb{H}^n$  in which case  $G = PO^+(n, 1)$ ,  $K = SO(n)$ ,  $M = SO(n-1)$  and  $K/M = S^{n-1}$ .) However,  $K$  still admits a product structure measurably, and so if we decompose  $G$  either locally (continuously) or globally (continuously off of a measure zero subset) as  $G = N \times A \times M \times K/M$  then the above discussion shows that we can describe all of the lifts of  $\mu$  which are left Haar measures on each orbit of  $N \rtimes A$  as

$$d\hat{\mu}(x, k, \xi, t) = d\mu_{x,\xi}(k) d\mu_t(\xi) dm_{N_\zeta \rtimes A_\zeta}^R(x) d\sigma(t),$$

where we have identified the support of  $\mu_t$  with  $K_p/M_p^\zeta = G/B$ .

If we wish to have  $\widehat{\mu}$  invariant under the right action of  $N_\zeta \rtimes A_\zeta$ , we first require that the measures  $\mu_{x,\xi}$  satisfy  $g_*\mu_{x,\xi} = \mu_{xg,\xi g} = \mu_{xg,\xi}$  where the initial push-forward is under the right action by  $g \in N_\zeta \rtimes A_\zeta$ . Second, we need that  $\mu_t\sigma$  is right invariant under  $N_\zeta \rtimes A_\zeta$ , but this follows from the fact that if  $t'$  is the transversal point corresponding to the plaque containing  $tg^{-1}$ , then  $\frac{d\mu_t}{dg_*\mu_t} = \frac{d\mu_t}{d\mu_{t'}}$  is the constant function on  $G/B$  with value  $\frac{dg_*\sigma}{d\sigma}(t)$ , as explained previously.

If we further require that  $\widehat{\mu}$  be invariant under the right action by  $M_p^\zeta$ , then since the  $\mu_t$  measures on  $K_p/M_p^\zeta$  are already right  $M_p^\zeta$  equivariant, one only needs to require that the measures  $\mu_{x,\xi}$  be right equivariant. However, the only right invariant measure on  $xk^\xi M_p^\zeta$  for any  $x, \xi$  is the (right=left) Haar measure,  $m_{M_p^\zeta}$ , so  $\mu_{x,\xi} = (xk^\xi)_* m_{M_p^\zeta}$ . Note that this is compatible with the right action of the group  $N_\zeta \rtimes A_\zeta$  which is transitive on  $x \in G/K$ .

However there is no problem to do the extension, since this right action, which is globally defined on the entire lamination  $\widehat{\mathcal{L}}$ , agrees on the overlap of two charts which have the same form in  $\widehat{\mathcal{L}}$  as in  $\mathcal{L}$ , except for the factor  $K$ . Moreover note that and therefore the action can be extended to all of  $\widehat{\mathcal{L}}$ . Hence there is a unique right  $B$ -invariant lift  $\widehat{\mu}$  of  $\mu$ . This finishes the proof of one direction of Theorem I.

From the above identifications, we immediately obtain the following corollary.

**Corollary 3.2.** *The  $N \rtimes A$  right invariant Borel measures  $\widehat{\mu}$  on  $\widehat{\mathcal{L}}$  are in one to one correspondence with measurable sections of the bundle  $E \rightarrow G/B$  whose fiber over  $\xi \in G/B$  is the space of Borel probability measures on  $M_{p,\xi}$ .*

**3.1. The Projection of an  $N \rtimes A$  Invariant Measure.** The purpose of this section is to prove the second half of the statement in the theorem.

**Proposition 3.3.** *The projection of a measure on  $\widehat{\mathcal{L}}$  which is right  $N \rtimes A$  invariant is harmonic.*

*Proof.* We may write a right  $N_\zeta \rtimes A_\zeta$ -invariant measure  $\widehat{\mu}$  as before as

$$d\widehat{\mu}(x, k, \xi, t) = d\mu_{x,\xi}(k) d\mu_t(\xi) dm_{N_\zeta \rtimes A_\zeta}^R(x) d\sigma(t),$$

with the additional requirement that the measures  $\mu_{x,\xi}$  satisfy  $g_*\mu_{x,\xi} = \mu_{xg,\xi}$  for all  $g \in \Gamma_L$ . Since the measures  $\sigma$  are preserved and the measures  $\mu_{x,\xi}$  are killed under the projection, it remains to verify that the projection is locally  $h \, d\text{vol}_L$  on the given plaque of the leaf  $L \in \mathcal{L}$ . However, this is precisely what was checked in Equation 3.1, but starting from the last line and working backwards.  $\square$

*Remark 3.4.* Note that it is not necessary that the measure  $\widehat{\mu}$  be  $N_\zeta \rtimes A_\zeta$  invariant in order for it to be a lift of a harmonic measure. However, it is a bit cumbersome to precisely describe the fiber over a harmonic measure if it does not possess the global invariance property.

This also completes the proof of Theorem I.

#### 4. THE LIFT OF A HARMONIC MEASURE

**4.1. Heat Kernels.** The laminated heat operator  $\mathcal{H} = \frac{d}{dt} - \Delta_{\mathcal{L}}$  has fundamental operator solution given by  $D_t = e^{t\Delta_{\mathcal{L}}}$ , which is to be interpreted as a formal power series in  $\Delta_{\mathcal{L}}$ . Note that the collection  $\{D_t\}_{t \geq 0}$  forms an abelian semigroup known as the *laminated heat semigroup*.

The heat diffusion along the leaves can be described in terms of the fundamental solution of the heat equation known as the *laminated heat kernel* and denoted by  $p_t : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{R}$  for  $t \in \times(0, +\infty)$ . It is given by

$$p_t(x, y) = \begin{cases} p_t^L(x, y), & \text{if } x, y \text{ belong to the same leaf } L \\ 0, & \text{otherwise,} \end{cases}$$

where  $p_t^L$  is the heat kernel on  $L$ . The laminated heat semigroup can be expressed as

$$D_t f(x) = \int_{L_x} p_t(x, y) f(y) dy,$$

for all  $f \in C^2(\mathcal{L})$ , where  $L_x$  is the leaf through  $x$  and  $dy$  represents the Riemannian volume measure on  $L_x$ .

A measure  $m$  on  $\mathcal{L}$  is *ergodic* with respect to the holonomy pseudogroup of the lamination if  $\mathcal{L}$  can not be partitioned into two measurable leaf-saturated subsets having positive  $m$  measure. Lucy Garnett ([Gar83]) proved the following Ergodic Theorem for harmonic measures, a nice interpretation of which can be found in [Can03].

**Theorem 4.1.** *Let  $\mu$  be any harmonic measure on  $\mathcal{L}$ . For  $\mu$ -almost every  $x \in \mathcal{L}$ , the limit of Krylov–Bogolyubov means*

$$\widetilde{\delta}_x := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=0}^{n-1} D_t \delta_x$$

*exists and is an ergodic harmonic measure. Moreover, every ergodic harmonic measure arises this way.*

*Remark 4.2.* By Theorem 7.3 of [Can03] we may instead define  $\widetilde{\delta}_x$  by continuous time averages  $\widetilde{\delta}_x := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T D_t \delta_x dt$ .

**4.2. The Borel action on the canonical  $K$ -bundle over  $\mathcal{L}$ .** Geometrically, we can realize the  $\mathcal{L}$  as a sub-bundle of the full tangential frame bundle of  $\mathcal{L}$  as follows. Let  $F\mathcal{L}$  be the bundle of all orthonormal frames of tangent vectors to the leaves of  $\mathcal{L}$ . In particular each fiber is homeomorphic to  $O(n)$ . Since an isometry is uniquely determined by its derivative action on a frame,  $G$  acts freely on  $F\tilde{\mathcal{L}}$  for each leaf  $L \in \mathcal{L}$ . The portion of the orbit of a single frame  $F \in F_x \tilde{\mathcal{L}}$  in the fiber over a point  $x \in \tilde{\mathcal{L}}$  is precisely  $K_x \cdot F$ , and in particular is homeomorphic to  $K_x$ . However, there are many such orbits in the fiber sitting over a single point of the leaf  $\tilde{\mathcal{L}}$ ; essentially,

there is one for each unit vector in a closed Weyl chamber modulo boundary identifications. In order to create a bundle on all of  $\mathcal{L}$  we need to choose a canonical section of reference fibers along a transversal. We do this as follows.

By transverse continuity (resp. transverse  $\nu$ -measurability) of the metric we may make a transversely continuous (resp. transversely measurable) identification of  $T_t L$  with  $\mathfrak{p}$  at each point  $t$  the transversal such that the set of regular vectors and singular vectors vary transversely in the same fashion. (Note that the entire singular set is determined by the metric alone and does not depend on the any choices like that of the Cartan subspace.) In particular the space of all Weyl chambers in  $\mathfrak{p}$  under the identification vary in the same fashion. Hence the action of  $G$  will be compatible with this choice of section in the sense that whenever  $t, t' \in T$  belong to the same leaf  $L$  and  $v \in T_t L$  and  $v' \in T_{t'} L$  are any two vectors which are identified to the same element of  $\mathfrak{p}$ , then the left invariant field of  $v'$  relative to the identification at  $t'$  has value at  $t$  which differs from  $v$  by an element of  $K_t$ . This follows from the fact that the action of  $G$  and in particular of  $\Gamma_L$  on left invariant fields of  $\tilde{L}$  acts at any point  $x$  by elements of  $K_x$ . Now we choose any frame  $F_o \in \mathfrak{p}$  and take its  $K$ -orbit. By the identification along the section we obtain a  $K_t$  fiber at each point  $t \in T$ . Taking  $G$ -orbits this extends analytically to each  $\tilde{L}$ . As explained above, this descends to a bundle over the quotient  $L$  under the action of  $\Gamma_L$ . By compatibility of the section, the resulting bundle is a transversely continuous (resp. measurable) sub-bundle of  $F\mathcal{L}$ . Thus we can realize  $\widehat{\mathcal{L}}$  as a sub-bundle of  $F\mathcal{L}$ .

Note that this construction does not give the stronger statement of the existence of a section  $\sigma : \mathcal{L} \rightarrow F\mathcal{L}$ . As in the case of a single leaf, existence of sections requires the vanishing of certain characteristic classes. As a simple example, this above construction also applies to a single semisimple group of compact type such as  $G = SO(n+1)$  in which case  $K = SO(n)$  and the  $K$  orbit is  $S^n = SO(n+1)/SO(n)$ . With  $n = 2$ , for instance, the absence of sections  $\sigma : S^2 \rightarrow T^1 S^2 \cong FS^2$  is well known.

The *laminated geodesic flow* is the flow  $\phi_t$  that, restricted to the unit tangent bundle of a leaf  $L$  of  $\mathcal{L}$ , coincides with the geodesic flow in  $T^1 L$ . If we identify the tangent space of  $[K] \in G/K$  with the polar subspace  $\mathfrak{p}$ , the orthogonal complement of  $\mathfrak{k}$  in the Lie algebra  $\mathfrak{g}$ , then we can write general elements of  $T^1 L = \Gamma \backslash T^1(G/K)$  as  $\Gamma g \cdot v$  for some  $g \in G$  and  $v \in \mathfrak{p}$ . Moreover, the geodesic flow is given by the map  $\Gamma g \cdot v \mapsto \Gamma g \exp(tv) \cdot v$ , see [BM00]. Except in the rank one case, we cannot identify  $T^1(G/K)$  as a homogeneous quotient of  $G$  itself, so we cannot describe this flow by any (single) right action of a one parameter subgroup  $T < G$ .

We can similarly define the *laminated Weyl chamber flow* on  $\tilde{\mathcal{L}}$  and the *laminated Borel action* on  $\widehat{\mathcal{L}}$  by combining the corresponding leafwise actions on  $\Gamma_L \backslash G/M$  and  $\Gamma_L \backslash G$ , respectively. These are given by global actions consisting of right multiplication by elements of  $A$  on  $\mathcal{L}$  and elements of

$B < G$  on  $\widehat{\mathcal{L}}$ , respectively. The standard unipotent action given by the restricted right action of  $N < B$  on  $\Gamma \backslash G$  generalizes the stable horocycle flow on  $\Gamma \backslash PSL(2, \mathbb{R}) = T^1(\Gamma \backslash \mathbb{H}^2)$ , for which case  $N \cong \mathbb{R}$ . Both the Weyl chamber flow on  $\widehat{\mathcal{L}}$  and the Borel action on  $\widehat{\mathcal{L}}$  have the same regularity as the full right  $G$ -action on  $\widehat{\mathcal{L}}$ .

**4.3. A measure invariant under the action of the affine group.** Let  $\pi : \widehat{\mathcal{L}} \rightarrow \mathcal{L}$  be the canonical projection, and consider a harmonic probability measure  $\mu$  on  $\mathcal{L}$ . In [Mar06], there is a construction in the case of  $G = SL(2, \mathbb{R})$  that produces a measure  $\widehat{\nu}$  on  $\widehat{\mathcal{L}}$  which is invariant under the stable (or the unstable) horocycle flow and such that  $\pi_* \widehat{\nu} = \mu$ . In this section we will construct  $\widehat{\nu}$  in a manner analogous to this construction, and it will follow from the properties that  $\widehat{\nu} = \widehat{\mu}$ .

We consider  $G$  as the particular sub-bundle of the full frame bundle mentioned earlier. We begin by using a fixed point  $p$ . As in [Mar06], two simplifying assumptions shall be made, that carry no loss of generality:

- (1) That  $\mu$  is ergodic; in particular, by Theorem 4.1 there is an  $p \in \mathcal{L}$  ( $\mu$ -almost any point will do) such that  $\mu$  arises as a limit of Krylov–Bogolyubov sums starting from an initial Dirac mass at  $p$ ; and
- (2) that the point  $p$  belongs to a leaf which is simply connected. (If it does not, we consider the universal cover of the leaf through  $p$ , as explained in [Mar06, pg.857].)

**4.4. The Construction of  $\widehat{\nu}$ .** For any natural number  $n \geq 1$ , let  $\delta_p^{(n)}$  be the measure given by the following Krylov–Bogolyubov sum,

$$\delta_p^{(n)} = \frac{1}{n} \sum_{t=0}^{n-1} D_t \delta_p.$$

More prosaically,  $\delta_p^{(n)}$  is the probability measure such that, for every continuous function  $f$  in  $\mathcal{L}$ ,

$$\int_{\mathcal{L}} f d\delta_p^{(n)} = \frac{1}{n} \sum_{t=0}^{n-1} D_t f(p) = \frac{1}{n} \sum_{t=0}^{n-1} \int_{L_p} p_t(p, y) f(y) dy.$$

With this notation, under our assumptions we have  $\mu = \lim_n \delta_p^{(n)}$  for a generic choice of  $p$ .

Consider the leaf  $L$  of  $\mathcal{L}$  passing through  $p$ . By our second assumption above,  $L$  is a global symmetric space identified with  $G/K_p$ . Let  $\mathcal{S} \subset L$  denote the (measure zero) set of points in  $L$  whose outward pointing radial vector from  $p$  is a singular vector in  $TL$  under this identification. By convention, we also assume  $p \in \mathcal{S}$ . In other words,  $\mathcal{S}$  consists of the union of points in the  $p$ -orbit of the chamber walls in all Cartan subgroups. Let  $\mathcal{V} : L \setminus \mathcal{S} \rightarrow \widehat{\mathcal{L}}$  be the “radial” Weyl chamber field defined as follows. For each point  $x \in L \setminus \mathcal{S}$  there is a unique unit speed geodesic passing through  $x$  starting from  $p$  that ends at a regular point  $\xi(x) \in \partial_\infty L$ , where here  $\partial_\infty L$  represents the geodesic



(ideal) boundary of  $L$ . Set  $\mathcal{V}(x) \in G/M \cong \check{L}$  to be the unique Weyl chamber that, when viewed as a subset of  $\mathfrak{g}$ , has exponential image in  $G/K \cong L$  with vertex at  $x$  and  $\xi(x)$  on its ideal boundary. We remark that  $\mathcal{V}$  does not extend continuously to all of  $L$ , but we can take a measurable extension which is continuous almost everywhere.

Now using the Cartan decomposition  $G = K_p \overline{A_\zeta^+} K_p$  we can identify points in  $\check{L} \cong G/M_p^\zeta$  with  $K_p \overline{A_\zeta^+} K_p / M_p^\zeta$  so that any Weyl chamber  $\mathcal{C}$  with base point over  $x = kaK_p \in G/K_p$  can be written as  $\mathcal{C} = kak'M_p^\zeta$ . (Note that  $k \cdot \xi = \xi(x)$ .) The defining condition for  $\mathcal{V}(x)$  is that  $k' = e$  in other words,  $\mathcal{V}(x) = kaM_p^\zeta$ . Since  $M_p^\zeta$  is the centralizer of  $A_\zeta$ ,  $a_o \in \overline{A_\zeta^+}$  acts on the right by  $\mathcal{V}(x)a_o = kaa_oM_p^\zeta = \mathcal{V}(xa_o)$  where  $xa_o$  means the point  $xa_o = kaa_oK_p$ . Here we have used that  $\overline{A_\zeta^+}$  is closed under multiplication. Consequently, the action of  $wa_o \in \overline{A_\zeta^+}$  sends the the section  $\mathcal{V}$  into, but not onto, itself. For since  $a_o, a \in \overline{A_\zeta^+}$ , the distance of the basepoint of the Weyl chamber to  $p$  is always increased by  $d(p, aa_o p) - d(p, ap) \geq 0$ , where  $d$  is the distance function on  $L$ .

This expansive action generalizes the positive geodesic flow acting on the unit radial vector field emanating from  $p$  in the rank one case. Note that it is well defined and continuous only because  $\mathcal{V}$  is defined on  $L \setminus \mathcal{S}$ . For convenience we shall extend  $\mathcal{V}$  to the measure zero set  $\mathcal{S}$  to obtain a measurable section  $\mathcal{V} : L \rightarrow \check{L}$ . Now that we have the frame field  $\mathcal{V}$ , we are ready to construct  $\hat{\nu}$ .

Define  $\mu_n = \mathcal{V}_* \delta_p^{(n)}$ , where recall we are treating  $\mathcal{V}$  as a mapping from  $L$  to  $\check{L}$ . This gives a sequence of probability measures on the compact space  $\check{\mathcal{L}}$ . Let  $\tilde{\nu}$  be any limit point of the sequence  $(\mu_n)$  in the sense of the weak-\* topology. (It is not difficult to see that in fact  $\tilde{\nu}$  is the unique limit of the  $\mu_n$ , but this is unessential for our argument and will follow anyhow from the properties after the fact.) Finally we define  $\hat{\nu}$  to be locally the product  $d\hat{\nu} = d\tilde{\nu} \times dm_M \times d\sigma$  where  $\sigma$ , as before, is the transversal measure of the original harmonic measure  $\mu$  and  $m_M$  represents the Haar measure on each of the fibers of the fibration  $\widehat{\mathcal{L}} \rightarrow \check{\mathcal{L}}$  which are canonically identified with  $M$ .

**Theorem 4.3.** *We have  $\hat{\nu} = \hat{\mu}$ .*

*Proof.* First note that  $\pi_* \hat{\nu} = \mu$  since  $\mathcal{V}$  is a section, and under our assumptions  $\mu = \lim_n \mu_n$ .

It remains to show that  $\hat{\nu}$  is invariant under the full right action by the minimal parabolic subgroup  $B = N_\zeta A_\zeta M_p^\zeta$ . The measure  $\hat{\nu}$  is right  $M_p^\zeta$  invariant since the Haar measures are, and the right action of this group is trivial on  $G/M$  and thus on  $\check{\nu}$ .

Next we show that  $\hat{\nu}$  is invariant under the right action by  $N_\zeta$ . We recall that  $\mathcal{V}$  is invariant under the left action by  $K_p$ . Observe that at a point  $g \in G$ , the  $K_p$  orbit  $K_p g$  has tangent space at  $g$  which is  $(R_g)_* \mathfrak{K}_p$  and

the pullback of this to the identity is  $(L_g)^*(R_g)_*\mathfrak{K}_p$  which is the tangent space at the identity to the conjugate stabilizer group  $K_{g^{-1}p}$ . This group acting on the left on  $p$  has orbit which is contained in the sphere of radius  $d(p, gp)$  in  $G/K$ . In fact this orbit is the entire sphere in rank one and has codimension one less than the rank in general. Setting  $A_\zeta^+ = \exp(\mathfrak{a}_\zeta^+)$ , take  $g = a_\zeta \in A_\zeta^+$ . As  $a_\zeta$  tends to infinity, which is to say that  $a_\zeta \rightarrow \xi \in \partial_\infty L$ , the spheres centered at  $gp$  of radius  $d(p, gp)$  limit to the horosphere based at  $p$  tangent to  $\xi$ . Both these orbits and the left orbit of  $N_\zeta$  admit the same transversals in the flat, but to show that the orbits converge one notes that the homogeneous spaces  $K_{g^{-1}p}/M_{g^{-1}p}^\zeta$  converge to  $N_\zeta$  since the quotient spaces of the subalgebras converge to the subalgebra  $\mathfrak{n}_\zeta$  in  $\mathfrak{g}$ . In other words  $a_\zeta^{-1}K_p a_\zeta$  tends to  $M_p^\zeta N_\zeta$ . In particular,  $a_\zeta^{-1}K_p a_\zeta M_p^\zeta = a_\zeta^{-1}K_p M_p^\zeta a_\zeta = a_\zeta^{-1}K_p a_\zeta$  tends to  $M_p^\zeta N_\zeta M_p^\zeta = M_p^\zeta N_\zeta$ . However, it is not the case that for any particular  $k \in K_p$  there is an  $n \in N_\zeta$  such that  $a_\zeta^{-1}ka_\zeta M_p^\zeta$  tends to  $M_p^\zeta n$  since the right  $N_\zeta$  action does not preserve  $M_p^\zeta$  cosets. (One can see this easily by noting the difference in the dimensions of the left orbit of  $p$  by these cosets in  $G/K$ .)

Since the heat kernel is  $G$ -equivariant,  $p_t(gp, hp) = p_t(p, g^{-1}hp)$ , and using the Cartan decomposition  $G = K_p \overline{A^+} K_p$  we may write  $p_t(p, kak'p) = p_t(p, ka \cdot p)$ . Moreover on a symmetric space  $p_t(p, \cdot)$  is symmetric under the action of  $K_p$ , so that  $p_t(p, ka \cdot p) = p_t(p, a \cdot p)$  (see [AJ99]). In other words, the heat kernel just depends on  $a \in A_\zeta^+$ . Hence the conditional probability measures  $\delta_p^{(n)}$  on any left  $K_p$  orbit in  $G/K$  are just the projections of the unit Haar measures on  $K_p$ . The lift under  $\mathcal{V}$  to  $G/M$  does not change this. In particular, the conditional measures for  $\mu_n$  on  $K_p \cdot a \cdot \mathcal{V}(p)$  do not depend on  $n$  and so we denote them by  $\alpha_a$ . We observe that these also coincide with the conditional probability measures of  $\mathcal{V}_* D_t \delta_p$  for all  $t \geq 0$ . In fact the  $\alpha_a$  do not really depend on  $a$  either, except to indicate the particular orbit they live on.

Since we lift these to  $G$  by the bi-invariant measure on  $M_p^\zeta$ , the radial conditional measures of  $\widehat{\nu}$  are the Haar measures on  $K_p$ . Thus their limit under conjugation by  $a_\zeta$  is also the bi-invariant measure on the space  $M_p^\zeta N_\zeta$ . This measure restricts to the (bi-invariant) Haar measures under both decompositions  $M_p^\zeta N_\zeta$  and  $N_\zeta M_p^\zeta$  of this space. In particular, these conditional measures are right  $N_\zeta$  invariant, and thus so is  $\widehat{\nu}$ .

To finish the proof of Theorem 4.3 we need to show invariance of  $\widehat{\nu}$  under the right action of  $A_\zeta^+$ . Since the right action by  $A_\zeta^+$  takes  $M_\zeta$  cosets to cosets, and preserves their Haar measures, it is enough to check the condition on  $\check{\nu}$ .

We will consider an arbitrary element  $a_o = \exp(H_o) \in A_\zeta^+$ . We need to show that  $(\phi_{a_o})_* \nu = \nu$  where  $\phi_{a_o}$  is simply the right action by  $a_o$ , namely  $R_{a_o}$  on  $G/M$ . (The purpose for the additional nomination  $\phi_{a_o}$  is merely to

suggest the geodesic flow which it generalizes.) For this it is sufficient to establish the following invariance under the heat flow.

For every continuous real-valued function  $f$  on  $\mathcal{Z}$  and every  $a_o \in A_\zeta^+$ ,

$$(*) \quad \lim_{t \rightarrow +\infty} \left| \int_{\mathcal{Z}} f d(\mathcal{V}_* D_t \delta_p) - \int_{\mathcal{Z}} f \circ \phi_{a_o} d(\mathcal{V}_* D_t \delta_p) \right| = 0.$$

Let  $S_a = K_p \cdot ap$  denote the  $K_p$  orbit of  $a \cdot p$ . Its normalized Lebesgue measure can be pushed forward by  $\mathcal{V}$  to obtain the measure  $\alpha_a$ , supported on the manifold  $\mathcal{V}(S_a) \subset \mathcal{Z}$ . However, since the section  $\mathcal{V}$  was constructed to be  $A_\zeta$  equivariant, we may reduce the verification of  $(*)$  to the corresponding condition on each conditional measure of  $\mu_n$ . To determine this condition, we first note that we can write the integral of  $f$  with respect to  $\mathcal{V}_* D_t \delta_p$  as

$$\int_{A_\zeta^+} u_t(a) \left( \int_{\mathcal{V}(S_a)} f d\alpha_a \right) dm_{A_\zeta}(a) = \int_{A_\zeta^+} u_t(a) \left( \int_{K_p} f(ka\mathcal{V}(p)) dm_{K_p}(k) \right) dm_{A_\zeta}(a),$$

where  $u_t(a) = p_t(p, a) \times \text{vol}(S_a)$  for any point  $a \in A_\zeta^+$  and  $dm_{K_p}$  is the probability Haar measure on  $K_p$ . (Recall that the heat kernel  $p_t(p, \cdot)$  only depends on  $a \in A_\zeta^+$ .) For  $a_o \in A_\zeta^+$ , the flow  $\phi_{a_o}$  takes  $\mathcal{V}(S_a)$  to  $\mathcal{V}(S_{aa_o})$ , and  $(\phi_{a_o})_* \alpha_a = \alpha_{aa_o}$ . Recall that the function  $f$  on  $\mathcal{Z}$  is necessarily bounded. Writing  $g(a) = \int_{K_p} f(k \cdot a \cdot \mathcal{V}(p)) dm_{K_p}(k)$ , and making use of the change of variables  $a \mapsto aa_o^{-1}$ , the condition  $(*)$  follows from:

For every continuous bounded function  $g : A_\zeta^+ \rightarrow \mathbb{R}$  and every  $a_o \in A_\zeta^+$ ,

$$(**) \quad \lim_{t \rightarrow \infty} \left| \int_{A_\zeta^+} u_t(a) g(a) dm_{A_\zeta}(a) - \int_{a_o A_\zeta^+} u_t(aa_o^{-1}) g(a) dm_{A_\zeta}(a) \right| = 0.$$

Note that  $A_\zeta^+$  may be decomposed, modulo measure zero sets, into the disjoint union  $\overline{A_\zeta^+} = (\cup_\alpha S_\alpha) \cup a_o A_\zeta^+$  where  $\alpha$  runs over the faces of a closed Weyl chamber and the  $S_\alpha$  are slabs of the form,

$$S_\alpha = \exp(\{sH_o + \mathfrak{F}_\alpha : s \in [0, 1]\})$$

where  $a_o = \exp H_o$  and  $\mathfrak{F}_\alpha \subset \partial \overline{\mathfrak{a}_\zeta^+}$  is a face of the Weyl chamber.

Applying this decomposition and translating the domain  $a_o A_\zeta^+$ , the equality  $(**)$  becomes:

$$(4.1) \quad \lim_{t \rightarrow \infty} \left| \int_{A_\zeta^+} (u_t(aa_o) - u_t(a)) g(aa_o) dm_{A_\zeta}(a) + \int_{\cup_\alpha S_\alpha} u_t(a) g(a) dm_{A_\zeta}(a) \right| = 0.$$

Finally this claim and the theorem, follow from the following proposition.  $\square$

**Proposition 4.4.** *For any  $a_o \in A_\zeta^+$ ,*

$$(4.2) \quad \lim_{t \rightarrow \infty} \int_{A_\zeta^+} |u_t(a) - u_t(aa_o)| dm_{A_\zeta}(a) = 0.$$

Moreover, the integral of  $u_t$  on any slab  $S_\alpha$  vanishes in the limit as  $t \rightarrow \infty$ .

*Proof.* We will express elements  $a \in A_\zeta^+$  additively by  $a = \exp(H)$  and  $a_o = \exp(H_0)$  for  $H, H_o \in \mathfrak{a}_\zeta^+$ . The heat kernel,  $p_t(p, gp)$  for  $g = nak$  only depends on  $a = \exp H$  so we write  $p_t(p, gp) = p_t(a)$ . Writing  $f \asymp g$  whenever there is a universal constant  $C \geq 1$  such that  $\frac{1}{C}g \leq f \leq Cg$ , the Main Theorem (3.1) of [AO03], states that

$$(4.3) \quad p_t(\exp(H)) \asymp t^{-\frac{\ell}{2}} \left( \prod_{\alpha \in \Lambda^{++}} \frac{1 + \langle \alpha, H \rangle}{t} \left( 1 + \frac{1 + \langle \alpha, H \rangle}{t} \right)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1} \right) e^{-|\rho|^2 t - \langle \rho, H \rangle - \frac{|H|^2}{4t}},$$

where  $\ell = \dim A$  is the rank of  $G$ ,  $\Lambda^{++} \subset \Lambda^+$  is the set of indecomposable positive roots,  $\rho$  is the algebraic centroid of  $\mathfrak{a}_\zeta^+$  defined by  $\rho = \frac{1}{2} \sum_{\alpha \in \Lambda^+} m_\alpha H_\alpha$  where  $H_\alpha$  are the root vectors with multiplicity  $m_\alpha = \dim \mathfrak{g}_\alpha$ .

Furthermore, one can obtain a sharp asymptotic estimate for  $p_t(\exp(H))$  when  $H$  stays away from chamber walls. More precisely, Proposition 3.2 of [AJ01] states that for any sequence  $H_j \in \mathfrak{a}^+$  such that  $\langle \alpha, H_j \rangle \rightarrow \infty$  for all  $\alpha \in \Lambda^+$  and any sequence  $t_j \rightarrow \infty$ , we have

$$(4.4) \quad \lim_{j \rightarrow \infty} \frac{1}{p_{t_j}(\exp H_j)} C_o t_j^{-\frac{\ell}{2}} \mathbf{c} \left( -\frac{iH_j}{2t_j} \right)^{-1} e^{-|\rho|^2 t_j - \langle \rho, H_j \rangle - \frac{|H_j|^2}{4t_j}} = 1,$$

where  $C_o > 0$  is a universal constant, and  $\mathbf{c}$  is the function of Harish-Chandra which in this range satisfies the bounds,

$$(4.5) \quad \mathbf{c} \left( -\frac{iH}{2t} \right)^{-1} \asymp \prod_{\alpha \in \Lambda^{++}} \frac{1 + \langle \alpha, H \rangle}{t} \left( 1 + \frac{\langle \alpha, H \rangle}{t} \right)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1}.$$

Moreover, the volume of the  $K_p$  orbit through  $\exp(H)$  is

$$C_1 \prod_{\alpha \in \Lambda^+} \sinh(\langle \alpha, H \rangle)^{m_\alpha}$$

for some constant  $C_1$  that depends on the normalization of the Haar measure used. As  $|H|$  increases this rapidly approaches  $C_1 e^{2\langle \rho, H \rangle}$ . So after multiplying  $p_t$  by the volume in the restricted regime, we obtain the asymptotic,

$$\begin{aligned} u_t(\exp H) &\sim C t^{-\frac{\ell}{2}} \mathbf{c} \left( -\frac{iH}{2t} \right)^{-1} e^{-|\rho|^2 t + \langle \rho, H \rangle - \frac{|H|^2}{4t}} = \\ &C t^{-\frac{\ell}{2}} \mathbf{c} \left( -\frac{iH}{2t} \right)^{-1} e^{-\frac{|H - 2\rho t|^2}{4t}}, \end{aligned}$$

where  $C > 0$  is the combined universal constant. Similarly from (4.3), in all cases we have the bounds,

$$u_t(\exp H) \asymp t^{-\frac{\ell}{2}} \left( \prod_{\alpha \in \Lambda^{++}} \frac{1 + \langle \alpha, H \rangle}{t} \left( 1 + \frac{1 + \langle \alpha, H \rangle}{t} \right)^{\frac{m_\alpha + m_{2\alpha}}{2} - 1} \right) e^{-\frac{|H - 2\rho t|^2}{4t}}.$$

This estimate shows that for  $|H|$  large and fixed,  $u_t(\exp H)$  is maximized when  $H$  is nearly in the direction of  $\rho$ . Moreover,  $u$  roughly decreases at least until  $H$  comes within a factor of  $\log |H|$  of the Weyl chamber walls.

Now denote by  $\mathfrak{w}(R) \subset \mathfrak{a}_\zeta^+$  the complement in  $\mathfrak{a}^+$  of the set  $\mathfrak{a}^+ + R\rho$ . Hence,  $W(R) = \exp(\mathfrak{w}(R))$  is the complement of the set  $\{\exp(H + R\rho) : H \in \mathfrak{a}^+\}$  and we set  $A(R) = A_\zeta^+ - W(R) = \exp(\mathfrak{a}^+ + R\rho)$ . From the definition of  $\rho$ , there are constants  $c_\alpha$  such that  $H$  belongs to  $\mathfrak{w}(R)$  if and only if  $\langle \alpha, H \rangle < c_\alpha R$  for some  $\alpha \in \Lambda^+$ .

Now we recall two important properties of the heat kernel, the heat kernel uniformly tends to zero on compacta as  $t \rightarrow \infty$ ):  $\int_{A_\zeta^+} u_t(a) dm_{A_\zeta}(a) = 1$  independent of  $t$ , and  $u_t$  tends to 0 uniformly on compacta (Theorem VIII.8 of [CRD84] and using that  $G/K$  is a Hadamard space). Since the ratio  $\frac{\text{vol } W(R) \cap B(x, r)}{\text{vol } A_\zeta^+ \cap B(x, r)}$  tends to 0 as  $r \rightarrow \infty$  for each fixed  $R > 0$ , the small scale average comparison above implies that,

$$\lim_{t \rightarrow \infty} \int_{W(R)} u_t(a) dm_{A_\zeta}(a) = 0.$$

Now we can compute

$$\begin{aligned} \int_{A(R)} |u_t(a) - u_t(aa_o)| dm_{A_\zeta}(a) &= \\ \int_{A(R)} u_t(a) \left| 1 - \frac{u_t(aa_o)}{u_t(a)} \right| dm_{A_\zeta}(a) &= \\ \int_{A_\zeta^+} u_t(\exp(H + R\rho)) \left| 1 - C(R) \frac{\mathbf{c}\left(-i\frac{H+H_o+R\rho}{2t}\right)^{-1} e^{-\frac{|H+H_o-(2t-R)\rho|^2}{4t}}}{\mathbf{c}\left(-\frac{iH+R\rho}{2t}\right)^{-1} e^{-\frac{|H-(2t-R)\rho|^2}{4t}}} \right| dm_{A_\zeta}(a = \exp(H)). \end{aligned}$$

Here  $C(R)$  tends to 1 as  $R \rightarrow \infty$ .

Now as shown in (4.5) above  $\mathbf{c}^{-1}$  has uniform polynomial growth in each  $\langle \alpha, H \rangle$  and hence for  $|H| \gg |H_o|$  and  $R \gg 0$  we have that

$$C(R) \frac{\mathbf{c}\left(-i\frac{H+H_o+R\rho}{2t}\right)^{-1}}{\mathbf{c}\left(-\frac{iH+R\rho}{2t}\right)^{-1}}$$

is very close to 1.

Putting this together and expanding the quadratic terms and making obvious cancelations yields,

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{A_\zeta^+} |u_t(a) - u_t(aa_o)| dm_{A_\zeta}(a) &= \\ \lim_{t \rightarrow \infty} \int_{A_\zeta^+} u_t(\exp(H)) \left| 1 - e^{-\frac{\langle H, H_o \rangle}{2t} + \langle \rho, H_o \rangle - \frac{|H_o|^2}{4t}} \right| dm_{A_\zeta}(\exp(H)) &= \\ \lim_{t \rightarrow \infty} \int_{A_\zeta^+} u_t(\exp(H)) \left| 1 - e^{-\frac{\langle H, H_o \rangle}{2t} + \langle \rho, H_o \rangle} \right| dm_{A_\zeta}(\exp(H)). \end{aligned}$$

Finally the Gaussian form of  $u_t$  implies that for  $t$  sufficiently large that there is a  $\delta(t)$ , tending to 0 as  $t \rightarrow \infty$ , and an  $\varepsilon(t)$  also tending to 0 as  $t \rightarrow \infty$  such that  $1 - \delta(t)$  of the mass of  $u_t(\exp(H))$  is contained in the  $\sqrt{t}^{1+\varepsilon(t)}$  ball centered at  $2\rho t$ , i.e.

$$|H - 2\rho t| < \sqrt{t}^{1+\varepsilon(t)}.$$

However, writing  $H = H' + 2\rho t$  we have

$$1 - e^{\langle \rho, H_o \rangle - \frac{\langle H, H_o \rangle}{2t}} = 1 - e^{-\frac{\langle H', H_o \rangle}{2t}}.$$

Furthermore, this function is on the order of at most  $\frac{1}{\sqrt{t}^{1-\varepsilon(t)}}$  on this ball.

Hence the limit of the integral vanishes since  $u_t$  is positive with unit mass.

For the last statement, note that the slab  $S_\alpha$  is the exponential image of a set contained within the Euclidean  $\delta$ -neighborhood of  $\mathfrak{F}_\alpha$  with width  $\delta$  at most  $|H_o|$ . On the other hand,  $H_o + \mathfrak{a}_\zeta^+ \subset \mathfrak{a}_\zeta^+$  is at finite Hausdorff distance from  $\mathfrak{a}_\zeta^+$ . Hence for any  $\varepsilon > 0$  and any sequence  $t_i \rightarrow \infty$  the exponential image of the ball of radius  $\sqrt{t_i}^{1+\varepsilon}$  centered at  $2\rho t_i$ , eventually does not even intersect  $S_\alpha$ . Since the integral of  $u_{t_i}$  on all of  $A_\zeta^+$  is one, and  $u_{t_i}$  is almost entirely supported on this ball, the value of  $u_t$  on  $S_\alpha$  vanishes as  $t \rightarrow \infty$ .  $\square$

## 5. APPLICATIONS AND RELATED RESULTS

As a simple application of Theorem I, we prove unique ergodicity for the action of  $B < G$  on the frame sub-bundle  $\widehat{\mathcal{L}}$  of certain foliations by locally symmetric spaces  $\Gamma_L \backslash G/K$ . We will need the following recent results on unique ergodicity for harmonic measures.

**5.1. Transversely conformal foliation by symmetric spaces.** A (transversely) conformal foliation of codimension  $q$  is essentially a foliation whose local holonomy maps are restrictions of conformal maps of the sphere  $S^q$ , where local transversals have been identified with subsets of the sphere. (For precise definitions see [Vai79].) This is a large class of foliations including all codimension one foliations and complex codimension one holomorphic foliations. In [DK07], Deroin and Kleptsyn prove the following:

**Theorem.** *Let  $(M, \mathcal{F})$  be a compact manifold together with a transversely conformal foliation and Riemannian metrics on the leaves that vary continuously in  $M$ . Let  $\mathcal{M}$  be a minimal set for  $\mathcal{F}$  (that is, a closed saturated subset of  $M$  where all leaves are dense.) If  $\mathcal{M}$  has no transverse holonomy-invariant measure, then it has a unique harmonic measure.*

Combining this with our main result immediately yields:

**Corollary 5.1.** *Let  $(M, \mathcal{F})$  be a compact manifold together with a transversely conformal foliation by locally symmetric spaces  $\Gamma_L \backslash G/K$ . Let  $\mathcal{M}$  be a minimal set for  $\mathcal{F}$ . If  $\mathcal{M}$  has no holonomy-invariant measures, then the action of  $B$  on  $\widehat{\mathcal{M}}$  is uniquely ergodic.*

This generalizes the corresponding result for surfaces in [BM08].

**5.2. Holonomy Invariant Measures.** We now give the proof of Theorem II, which we state again for convenience.

**Theorem 5.2** ([MM09]). *There is a natural bijective correspondence between  $G$ -invariant measures on  $\widehat{\mathcal{L}}$  and invariant transverse measures on  $\mathcal{L}$ .*

*Proof.* We will establish a bijection  $\Phi$  from the holonomy invariant harmonic measures to right  $G$ -invariant measures. For a measure  $\mu$  on transversals of  $\widehat{\mathcal{L}}$  which is holonomy invariant, we set  $\Phi(\mu)$  to simply be the product of the (holonomy invariant) transverse conditional measures of  $\mu$  with the image of the left Haar measure under the discrete action of  $\Gamma_L$  on the universal cover of leaves of  $\widehat{L}$  identified with  $G$ . Since  $G$  is unimodular, these measures are the same as the right Haar measures on  $G$ . Hence the global measure  $\Phi(\mu)$  is right  $G$ -invariant.

For the converse we use the identification of the various definitions of holonomy invariance given by A. Connes in [Con82]. A holonomy invariant measure can be thought of an equivalence class of objects of the form  $[\omega, \mu]$  where  $\omega$  is a volume form on the leaves and  $\mu$  is a measure on  $\widehat{\mathcal{L}}$  which is  $\omega$ -invariant. If  $\mu$  is a  $G$ -invariant measure on  $\widehat{\mathcal{L}}$  and  $\omega$  is the (left=right) Haar measure on the leaves, the pair  $[\omega, \mu]$  determines a holonomy-invariant measure on transversals of  $\widehat{\mathcal{L}}$  (and therefore of  $\mathcal{L}$ .)  $\square$

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